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Existence of global weak solutions for Navier–Stokes equations with large flux ☆

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ABSTRACT

Global existence of weak solutions to the Navier–Stokes equations in a cylindrical domain under boundary slip conditions and with inflow and outflow is proved. To prove the energy estimate, crucial for the proof, we use the Hopf function. This makes it possible to derive an estimate such that the inflow and outflow need not vanish as $t \to \infty$. The proof requires estimates in weighted Sobolev spaces for solutions to the Poisson equation. Our result is the first step towards proving the existence of global regular special solutions to the Navier–Stokes equations with inflow and outflow.

1. Introduction

We consider viscous incompressible fluid motion in a finite cylinder with large inflow and outflow, assuming boundary slip conditions. Hence, the following initial boundary value problem is examined:

$$
\begin{aligned}
& v_t + v \cdot \nabla v - \operatorname{div} \mathbb{T}(v, p) = f \quad \text{in } \Omega^T = \Omega \times (0, T), \\
& \operatorname{div} v = 0 \quad \text{in } \Omega^T,
\end{aligned}
$$

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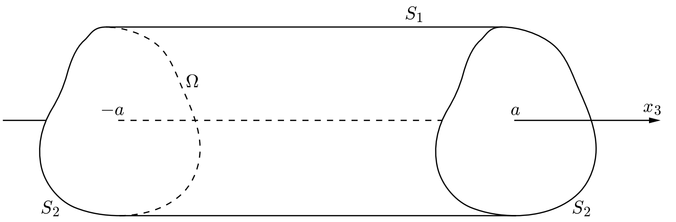

Fig. 1. Domain Ω.

$$
\begin{aligned}
v\cdot\bar n &= 0 \quad \text{on } S_1^T,\\
\nu\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha+\gamma v\cdot\bar\tau_\alpha &= 0,\quad \alpha=1,2 \text{ on } S_1^T,\\
v\cdot\bar n &= d \quad \text{on } S_2^T,\\
\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha &= 0,\quad \alpha=1,2 \text{ on } S_2^T,\\
v|_{t=0} &= v(0) \quad \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $\Omega\subset\mathbb{R}^3$ is a cylindrical domain, $S=\partial\Omega$, v is the velocity of the fluid motion with $v(x,t)=(v_1(x,t),v_2(x,t),v_3(x,t))\in\mathbb{R}^3$, $p=p(x,t)\in\mathbb{R}^1$ denotes the pressure, $f=f(x,t)=(f_1(x,t),f_2(x,t),f_3(x,t))\in\mathbb{R}^3$ is the external force field, $x=(x_1,x_2,x_3)$ are the Cartesian coordinates, $\bar n$ is the unit outward vector normal to the boundary S and $\bar\tau_\alpha$, $\alpha=1,2$, are tangent vectors to S and $\cdot$ denotes the scalar product in $\mathbb{R}^3$. We define the stress tensor $\mathbb{T}(v,p)$ as

$$\mathbb{T}(v,p)=\nu\mathbb{D}(v)-p\mathbb{I},$$

where ν is the constant viscosity coefficient and $\mathbb{I}$ is the unit matrix. Next, $\gamma>0$ is the slip coefficient and $\mathbb{D}(v)$ denotes the dilatation tensor of the form

$$\mathbb{D}(v)=\{v_{i,x_j}+v_{j,x_i}\}_{i,j=1,2,3}.$$

We assume that $\Omega\subset\mathbb{R}^3$ is a cylindrical type domain parallel to the x_3 axis with arbitrary cross section (see Fig. 1). We set $S=S_1\cup S_2$ where S_1 is the part of the boundary which is parallel to the x_3 axis and S_2 is perpendicular to it. Hence

$$
\begin{aligned}
S_1 &= \left\{x\in\mathbb{R}^3\colon \varphi_0(x_1,x_2)=c_0,\ -a<x_3<a\right\},\\
S_2(-a) &= \left\{x\in\mathbb{R}^3\colon \varphi_0(x_1,x_2)<c_0,\ x_3=-a\right\},\\
S_2(a) &= \left\{x\in\mathbb{R}^3\colon \varphi_0(x_1,x_2)<c_0,\ x_3=a\right\}
\end{aligned}
$$

where a, c_0 are given positive numbers and $\varphi_0(x_1,x_2)=c_0$ describes a sufficiently smooth closed curve in the plane $x_3=\text{const}$.

To describe the inflow and outflow we define

$$
\begin{aligned}
d_1 &= -v\cdot\bar n|_{S_2(-a)},\\
d_2 &= v\cdot\bar n|_{S_2(a)},
\end{aligned}
\tag{1.2}
$$

with $d_i \geqslant 0$, $i = 1, 2$. We require the compatibility condition

$$\int\limits_{S_2(-a)} d_1\, dS_2 = \int\limits_{S_2(a)} d_2\, dS_2. \tag{1.3}$$

The aim of this paper is to prove the existence of global weak solutions to problem (1.1) without restrictions on the magnitudes of the external force f, initial data $v(0)$, inflow d_1 and outflow d_2. We would like to show the existence of solutions such that the flux does not have to vanish as $t \to \infty$. The main result of this paper is the starting point for showing existence of global regular special solutions to the Navier Stokes equations with some assumptions on data only in order to omit the so-called sufficient conditions which are now very popular. We underline, that our restrictions admit much more general class of solutions than in [3,4,9] because in these papers the flux must converge to zero sufficiently fast or there is no flux. The improvement in the paper is possible by applying the Hopf function (see [5]) and weighted estimates proved in [6,7].

We define a space natural for the study of weak solutions to the Navier–Stokes equations:

$$V_2^0(\Omega^T) = \left\{ u\colon \|u\|_{V_2^0(\Omega^T)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|u\|_{L_2(\Omega)} + \left(\int\limits_0^T \|\nabla u\|^2_{L_2(\Omega)}\, dt \right)^{1/2} < \infty \right\}.$$

To simplify notation, we do not distinguish between norms of scalar and vector functions and we write

$$\|f\| := \sum_{i=1}^3 \|f_i\| \quad \text{for any } f = (f_1, f_2, f_3).$$

We also use

$$\|d\| := \|d_1\| + \|d_2\|$$

for the inflow d_1 and outflow d_2.

Theorem 1. *Assume the compatibility condition* (1.3). *Assume that $v(0) \in L_2(\Omega)$; $f \in L_2(0, T; L_{6/5}(\Omega))$; $d_i \in L_\infty(0, T; W_p^{s-1/p}(S_2)) \cap L_2(0, T; W_2^{1/2}(S_2))$; $\frac{3}{p} + \frac{1}{3} \leqslant s$, $p > 3$ or $p = 3$, $s > \frac{4}{3}$; and $d_{i,t} \in L_2(0, T; W_{6/5}^{1/6}(S_2))$, $i = 1, 2$. Then there exists a weak solution v to problem* (1.1) *such that v is weakly continuous with respect to t in $L^2(\Omega)$ norm and v converges to v_0 as $t \to 0$ strongly in $L^2(\Omega)$ norm. Moreover, $v \in V_2^0(\Omega^T)$, $v \cdot \bar\tau_\alpha \in L_2(0, T; L_2(S_1))$, $\alpha = 1, 2$, and v satisfies, for all $t \leqslant T$*

$$\begin{aligned} &\|v\|^2_{V_2^0(\Omega^t)} + \gamma \sum_{\alpha=1}^2 \int\limits_0^t \|v \cdot \bar\tau_\alpha\|^2_{L_2(S_1)} \\ &\quad\leqslant 2\|f\|^2_{L_2(0,t;L_{6/5}(\Omega))} + \varphi\Big(\sup_{\tau\leqslant t} \|d\|_{W_3^{s-1/p}(S_2)} \Big) \big(\|d\|^2_{L_2(0,t;W_2^{1/2}(S_2))} + \|d_t\|^2_{L_2(0,t;W_{6/5}^{1/6}(S_2))} \big) \\ &\qquad + \big\| v(0) \big\|^2_{L_2(\Omega)} \end{aligned} \tag{1.4}$$

where φ is a nonlinear positive increasing function.

Theorem 2. *Assume the compatibility condition* (1.3)*. Let* $f \in L_2(kT,(k+1)T;L_{6/5}(\Omega))$*,* $d_i \in L_\infty(\mathbb{R}^+;W_p^{s-1/p}(S_2)) \cap L_2(kT,(k+1)T;W_2^{1/2}(S_2))$*, where* $\frac{3}{p}+\frac{1}{3} \leqslant s$*,* $p>3$ *or* $p=3$*,* $s>\frac{4}{3}$*, and* $d_{i,t} \in L_2(kT,(k+1)T;W_{6/5}^{1/6}(S_2))$*,* $i=1,2$*. Let us assume that*

$$\|v(0)\|_{L_2(\Omega)} \leqslant A$$

for some constant A *and*

$$2\int_{kT}^{(k+1)T} \|f\|_{L_{6/5}(\Omega)}^2 + \varphi\Big(\sup_t \|d\|_{W_p^{s-1/p}(S_2)}\Big) \int_{kT}^{(k+1)T} \big(\|d\|_{W_2^{1/2}(S_2)}^2 + \|d_t\|_{W_{6/5}^{1/6}(S_2)}^2\big) \leqslant \big(1-e^{-\nu T}\big)A^2$$

for all $k \in \mathbb{N}_0$*, where* φ *is a nonlinear positive increasing function. Then there exists a global weak solution* v *to* (1.1) *such that*

$$v \in V_2^0\big(\Omega \times \big(kT,(k+1)T\big)\big), \quad \forall k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

and

$$\begin{aligned}\|v\|_{V_2^0(\Omega\times(kT,t))}^2 \leqslant {} & 2\int_{kT}^{t} \|f\|_{L_{6/5}(\Omega)}^2\,d\tau + A^2 \\ & + \varphi\Big(\sup_\tau \|d\|_{W_p^{s-1/p}(S_2)}\Big)\int_{kT}^{t} \big(\|d\|_{W_2^{1/2}(S_2)}^2 + \|d_t\|_{W_{6/5}^{1/6}(S_2)}^2\big)\,d\tau \end{aligned} \tag{1.5}$$

for $t \in (kT,(k+1)T]$*.*

The main step in this proof is estimate (2.7) – see Lemma 2.2. To derive it, we use the Hopf function (see [5,2]) and estimates in weighted Sobolev spaces (see [6,7]). This estimate enables us such to prove the global estimate (4.4) and to obtain global existence without the assumption of vanishing of the inflow–outflow and the external force. The paper generalizes the result from [8] to the inflow–outflow case. On the other hand, the existence of global regular special solutions in the case without flux was proved in [10].

2. Estimates

To show the existence theorem, we need to obtain an energy type estimate, and for this purpose, we have to make the Neumann boundary condition $(1.1)_5$ homogeneous.

To this end, we extend the functions corresponding to the inflow and outflow so that

$$\tilde{d}_i|_{S_2(a_i)} = d_i, \quad i=1,2, \qquad a_1=-a, \qquad a_2=a. \tag{2.1}$$

We introduce the function η (see [5]).

$$\eta(\sigma;\varepsilon,\rho) = \begin{cases} 1, & 0 \leqslant \sigma \leqslant \rho e^{-1/\varepsilon} \equiv r, \\ -\varepsilon \ln\frac{\sigma}{\rho}, & r < \sigma \leqslant \rho, \\ 0, & \rho < \sigma < \infty. \end{cases}$$

We calculate

$$\frac{d\eta}{d\sigma} = \eta'(\sigma;\varepsilon,\rho) = \begin{cases} 0, & 0 < \sigma \leqslant r, \\ -\frac{\varepsilon}{\sigma}, & r < \sigma \leqslant \rho, \\ 0, & \rho < \sigma < \infty, \end{cases}$$

so that $|\eta'(\sigma;\varepsilon,\rho)| \leqslant \frac{\varepsilon}{\sigma}$. We define functions η_i on the neighborhood of S_2 (inside Ω) by setting:

$$\eta_i = \eta(\sigma_i;\varepsilon,\rho), \quad i = 1, 2,$$

where σ_i denote local coordinates defined on a small neighborhood of $S_2(a_i)$:

$$\sigma_1 = a + x_3, \qquad \sigma_2 = a - x_3$$

and we set

$$\alpha = \sum_{i=1}^{2} \tilde{d}_i \eta_i,$$
$$b = \alpha \bar{e}_3, \qquad \bar{e}_3 = (0,0,1). \tag{2.2}$$

We set

$$u = v - b. \tag{2.3}$$

Therefore,

$$\operatorname{div} u = -\operatorname{div} b = -\alpha_{x_3} \quad \text{in } \Omega,$$
$$u \cdot \bar{n} = 0 \quad \text{on } S.$$

Thus, the boundary condition for u is homogeneous. The compatibility condition takes the form

$$\int_{\Omega} \alpha_{,x_3}\, dx = -\int_{S_2(-a)} \alpha|_{x_3=-a}\, dS_2 + \int_{S_2(a)} \alpha|_{x_3=a}\, dS_2 = 0.$$

We define φ as a solution to the Neumann problem

$$\Delta\varphi = -\operatorname{div} b \quad \text{in } \Omega,$$
$$\bar{n} \cdot \nabla\varphi = 0 \quad \text{on } S,$$
$$\int_{\Omega} \varphi\, dx = 0. \tag{2.4}$$

Next, we set

$$w = u - \nabla\varphi = v - (b + \nabla\varphi) \equiv v - \delta. \tag{2.5}$$

Consequently, (w, p) is a solution to the following problem:

$$
\begin{aligned}
& w_t + w\cdot\nabla w + w\cdot\nabla\delta + \delta\cdot\nabla w - \operatorname{div}\mathbb{T}(w,p) \\
&\quad = f - \delta_t - \delta\cdot\nabla\delta + \nu\operatorname{div}\mathbb{D}(\delta) = F(\delta,t) \quad \text{in } \Omega^T, \\
&\operatorname{div} w = 0 \quad \text{in } \Omega^T, \\
&w\cdot\bar n = 0 \quad \text{on } S^T, \\
&\nu\bar n\cdot\mathbb{D}(w)\cdot\bar\tau_\alpha + \gamma w\cdot\bar\tau_\alpha = -\nu\bar n\cdot\mathbb{D}(\delta)\cdot\bar\tau_\alpha - \gamma\delta\cdot\bar\tau_\alpha = B_{1\alpha}(\delta), \quad \alpha = 1,2 \text{ on } S_1^T, \\
&\bar n\cdot\mathbb{D}(w)\cdot\bar\tau_\alpha = -\bar n\cdot\mathbb{D}(\delta)\cdot\bar\tau_\alpha = B_{2\alpha}(\delta), \quad \alpha = 1,2 \text{ on } S_2^T, \\
&w|_{t=0} = v(0) - \delta(0) = w(0) \quad \text{in } \Omega,
\end{aligned} \tag{2.6}
$$

where $\operatorname{div}\delta = 0$. Moreover, we set

$$
\bar n|_{S_1} = \frac{(\varphi_{,x_1}, \varphi_{,x_2}, 0)}{\sqrt{\varphi_{,x_1}^2 + \varphi_{,x_2}^2}}, \qquad \bar\tau_1|_{S_1} = \frac{(-\varphi_{,x_2}, \varphi_{,x_1}, 0)}{\sqrt{\varphi_{,x_1}^2 + \varphi_{,x_2}^2}}, \qquad \bar\tau_2|_{S_1} = (0,0,1) = \bar e_3,
$$
$$
\bar n|_{S_2(-a)} = -\bar e_3, \qquad \bar n|_{S_2(a)} = \bar e_3, \qquad \bar\tau_1|_{S_2} = \bar e_1, \qquad \bar\tau_2|_{S_2} = \bar e_2
$$

where $\bar e_1 = (1,0,0)$, $\bar e_2 = (0,1,0)$.

We define a weak solution to problem (2.6):

Definition 2.1. We call w a weak solution to problem (2.6) if for any sufficiently smooth function ψ such that

$$
\operatorname{div}\psi|_\Omega = 0, \qquad \psi\cdot\bar n|_S = 0
$$

the integral equality

$$
\begin{aligned}
&\int_{\Omega^T} w_t\cdot\psi\,dx\,dt + \int_{\Omega^T} H(w)\cdot\psi\,dx\,dt + \nu\int_{\Omega^T}\mathbb{D}(v)\cdot\mathbb{D}(\psi)\,dx\,dt \\
&\quad + \gamma\sum_{\alpha=1}^2\int_{S_1^T} w\cdot\bar\tau_\alpha\psi\cdot\bar\tau_\alpha\,dS_1\,dt - \sum_{\alpha,\sigma=1}^2\int_{S_\sigma^T} B_{\sigma\alpha}\psi\cdot\bar\tau_\alpha\,dS_\sigma\,dt = \int_{\Omega^T} F\cdot\psi\,dx\,dt
\end{aligned}
$$

holds, where

$$
H(w) = w\cdot\nabla w + w\cdot\nabla\delta + \delta\cdot\nabla w.
$$

Lemma 2.2. *Assume the compatibility condition* (1.3). *Assume that* $f \in L_2(0,T;L_{6/5}(\Omega))$, $d_i \in L_\infty(0,T;W_p^{s-1/p}(S_2)) \cap L_2(0,T;W_2^{1/2}(S_2))$, *where* $\frac{3}{p} + \frac{1}{3} \leqslant s$, $p > 3$ *or* $p = 3$, $s > \frac{4}{3}$, $d_{i,t} \in L_2(0,T;W_{6/5}^{1/6}(S_2))$, $i = 1,2$, $w(0) \in L_2(\Omega)$. *Then every weak solution to* (2.6) *satisfies, for all* $t \leqslant T$,

$$
\begin{aligned}
&\|w\|^2_{V_2^0(\Omega^t)} + \gamma\sum_{\alpha=1}^2\int_0^t\|w\cdot\bar\tau_\alpha\|^2_{L_2(S_1)} \\
&\quad \leqslant 2\|f\|^2_{L_2(0,t;L_{6/5}(\Omega))} + \varphi\Big(\sup_\tau\|d\|_{W_p^{s-1/p}(S_2)}\Big)\big(\|d\|^2_{L_2(0,t;W_2^{1/2}(S_2))} + \|d_t\|^2_{L_2(0,t;W_{6/5}^{1/6}(S_2))}\big) \\
&\qquad + \big\|w(0)\big\|^2_{L_2(\Omega)}
\end{aligned} \tag{2.7}
$$

where $d = (d_1, d_2)$ *and* φ *is a nonlinear positive increasing function.*

Proof. We use $\psi = w$ as a test function in Definition 2.1 and apply the definition of F to obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L_2(\Omega)} + \int_\Omega (w\cdot\nabla\delta\cdot w + \delta\cdot\nabla w\cdot w)\,dx - \int_\Omega \operatorname{div}\mathbb{T}(w+\delta,p)\cdot w\,dx$$
$$= \int_\Omega (f - \delta_t - \delta\cdot\nabla\delta)\cdot w\,dx.$$

We use the boundary conditions on S_1 and S_2 in (1.1) to reformulate the third integral on the l.h.s. of the above inequality as follows

$$\begin{aligned}
\int_\Omega \operatorname{div}\mathbb{T}(w+\delta,p)\cdot w\,dx &= \int_\Omega \operatorname{div}\big[\nu\mathbb{D}(w+\delta) - p\mathbb{I}\big]\cdot w\,dx \\
&= \int_\Omega \operatorname{div}\big[\nu\mathbb{D}(w+\delta)\big]\cdot w\,dx - \int_\Omega p\cdot\nabla w \\
&= \int_\Omega D_{ij}(w+\delta)w_{j,x_i}\,dx = \int_\Omega D_{ij}(w)w_{j,x_i}\,dx + \int_\Omega D_{ij}(\delta)w_{j,x_i}\,dx \\
&= \frac{1}{2}\int_\Omega \big|D_{ij}(w)\big|^2\,dx + \int_\Omega D_{ij}(\delta)w_{j,x_i}\,dx.
\end{aligned}$$

Then, we apply the Korn inequality to obtain the estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w\|^2_{L_2(\Omega)} + \nu\|w\|^2_{H^1(\Omega)} + \gamma\sum_{\alpha=1}^2\|w\cdot\bar\tau_\alpha\|^2_{L_2(S_1)} \\
&\quad\leqslant -\int_\Omega (w\cdot\nabla\delta\cdot w + \delta\cdot\nabla w\cdot w)\,dx + c\sum_{\alpha=1}^2\|\delta\cdot\bar\tau_\alpha\|^2_{L_2(S_1)} \\
&\qquad + c\big\|\mathbb{D}(\delta)\big\|^2_{L_2(\Omega)} + \int_\Omega (f-\delta_t-\delta\cdot\nabla\delta)w\,dx.
\end{aligned}\tag{2.8}$$

Next, we focus on the integral

$$\begin{aligned}
\int_\Omega \delta\cdot\nabla w\cdot w\,dx &= \int_\Omega (b+\nabla\varphi)\cdot\nabla w\cdot w\,dx \\
&= \int_\Omega b\cdot\nabla w\cdot w\,dx + \int_\Omega \nabla\varphi\cdot\underline{w}\cdot w\,dx = I_1 + I_2.
\end{aligned}$$

We can estimate I_1 by the Hölder inequality and the definition of b:

$$\begin{aligned}
|I_1| &\leqslant \|\nabla w\|_{L_2(\Omega)}\|w\|_{L_6(\Omega)}\|b\|_{L_3(\Omega)} \leqslant c\|w\|^2_{H^1(\Omega)}\|b\|_{L_3(\widetilde S_2(\rho))} \\
&\leqslant c\rho^{1/6}\|w\|^2_{H^1(\Omega)}\|b\|_{L_6(\widetilde S_2(\rho))} \leqslant c\rho^{1/6}\|w\|^2_{H^1(\Omega)}\|\delta\|_{L_6(\Omega)} \\
&\leqslant c\rho^{1/6}\|w\|^2_{H^1(\Omega)}\|\tilde d\|_{H^1(\Omega)}
\end{aligned}$$

where

$$\widetilde{S}_2(\rho) = \{x \in \Omega\colon x_3 \in (-a, -a+\rho) \cup (a-\rho, a)\} = \widetilde{S}_2(\rho, a_1) \cup \widetilde{S}_2(\rho, a_2).$$

We estimate I_2 as follows

$$|I_2| = \left| \int_\Omega \nabla\varphi \cdot \nabla w \cdot w \, dx \right| \leqslant \|\nabla\varphi\|_{L_3(\Omega)} \|w\|_{L_6(\Omega)} \|\nabla w\|_{L_2(\Omega)} \tag{2.9}$$

where

$$\begin{aligned} \|\nabla\varphi\|_{L_3(\Omega)} &\leqslant c\|\nabla\varphi\|_{L_{3,-\mu'}(\Omega)} \leqslant c\|\nabla_{x_3}\nabla\varphi\|_{L_{3,1-\mu'}(\Omega)} \leqslant c\|\varphi\|_{L^2_{3,1-\mu'}(\Omega)} \\ &\leqslant c\|\operatorname{div} b\|_{L_{3,1-\mu'}(\Omega)} \end{aligned}$$

and we denote

$$\|u\|_{L^k_{p,\mu}(\Omega)} = \left(\sum_{|\alpha|=k} \int \left|D^\alpha_x u\right|^p_{\min_{i=1,2}} \left|\operatorname{dist}\left(x, S_2(a_i)\right)\right|^{p\mu} dx \right)^{1/p}, \quad \mu \in \mathbb{R},\ p \in (1, \infty).$$

To estimate the last norm, we use the result of [7] on the Poisson equation in weighted Sobolev spaces and choose $\frac{2}{3} \leqslant 1-\mu' \leqslant 1$. With $\mu = 1-\mu'$ we have

$$\begin{aligned} c\|\operatorname{div} b\|_{L_{3,\mu}(\Omega)} &\leqslant c\varepsilon \left(\sum_{i=1}^2 \int_{\widetilde{S}_2(a_i)} |\tilde{d}_i|^3 \frac{\sigma_i^{3\mu}}{\sigma_i^3} \, dx \right)^{1/3} + \left(\sum_{i=1}^2 \int_{\widetilde{S}_2(a_i)} |\tilde{d}_{i,x_3}|^3 \left|\rho(x)\right|^{3\mu} dx \right)^{1/3} \\ &\leqslant c \sum_{i=1}^2 \varepsilon \left(\sup_{x_3} \int_{S_2(a_i)} |\tilde{d}_i|^3 \, dx' \int_r^\rho \frac{\sigma_i^{3\mu}}{\sigma_i^3} \, d\sigma_i \right)^{1/3} \\ &\quad + \sum_{i=1}^2 \left(\sup_{x_3} \int_{S_2(a_i)} |\tilde{d}_{i,x_3}|^3 \, dx' \int_0^\rho \sigma_i^{3\mu} \, d\sigma_i \right)^{1/3} \\ &\leqslant c\varepsilon\rho^{\mu-2/3} \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + c\rho^{\mu+1/3} \sup_{x_3} \|\tilde{d},_{x_3}\|_{L_3(S_2)} \end{aligned}$$

where $\sigma_i = \operatorname{dist}\{S_2(a_i), x\}$, $x \in S_2(a_i, \rho)$. We note that the last bound holds for $\mu > \frac{2}{3}$ since for $\mu = \frac{2}{3}$ the r.h.s. takes the form

$$c \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + c\rho \sup_{x_3} \|\tilde{d},_{x_3}\|_{L_3(S_2)},$$

which cannot be made small for large $\tilde{d}$. Then,

$$|I_2| \leqslant c\left[\varepsilon\rho^{\mu-2/3} \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + \rho^{\mu+1/3} \sup_{x_3} \|\tilde{d},_{x_3}\|_{L_3(S_2)}\right] \|w\|^2_{H^1(\Omega)}.$$

Next, we consider the term

$$\int_\Omega (w\cdot\nabla\delta\cdot w)\,dx=\int_\Omega (w\cdot\nabla b\cdot w)\,dx+\int_\Omega (w\cdot\nabla\nabla\varphi\cdot w)\,dx=I_3+I_4.$$

For I_4, we have

$$\begin{aligned}|I_4|&\leqslant\left|\int_\Omega \operatorname{div}(w\cdot\nabla\varphi\cdot w)\,dx-\int_\Omega (w\cdot\nabla w\cdot\nabla\varphi)\,dx\right|\\&\leqslant\int_S\left|\bar n\cdot\nabla\varphi\cdot w^2\right|dS+\int_\Omega\left|\nabla\varphi\cdot(w\cdot\nabla w)\right|dx\leqslant\int_\Omega\left|\nabla\varphi\cdot(w\cdot\nabla w)\right|dx\end{aligned}$$

so I_4 can be treated in the same way as I_2 and therefore

$$|I_4|\leqslant c\Big[\varepsilon\rho^{\mu-2/3}\sup_{x_3}\|\tilde d\|_{L_3(S_2)}+\rho^{\mu+1/3}\sup_{x_3}\|\tilde d_{,x_3}\|_{L_3(S_2)}\Big]\|w\|^2_{H^1(\Omega)}. \tag{2.10}$$

On the other hand, using $b=\alpha\bar e_3=\sum_{i=1}^2\tilde d_i\eta_i\bar e_3$, we find a bound for I_3:

$$\begin{aligned}|I_3|&\leqslant\left|\sum_{i=1}^2\int_{\tilde S_2(\rho,a_i)} w\cdot\nabla(\tilde d_i\eta_i)w_3\,dx\right|\\&\leqslant\left|\sum_{i=1}^2\int_{\tilde S_2(\rho,a_i)}(w\cdot\nabla\tilde d_i\eta_i w_3+w\cdot\nabla\eta_i\tilde d_i w_3)\,dx\right|\\&\leqslant\sum_{i=1}^2\left(\int_{\tilde S_2(\rho,a_i)}|w\cdot\nabla\tilde d_i\eta_i||w_3|\,dx+\int_{\tilde S_2(\rho,a_i)}\varepsilon\left|\frac{w_3}{\sigma_i}w_3\tilde d_i\right|d\sigma_i\,dx_1\,dx_2\right)\\&\leqslant c\sum_{i=1}^2\|w\|_{L_6(\tilde S_2(\rho,a_i))}\|w_3\|_{L_3(\tilde S_2(\rho,a_i))}\|\nabla\tilde d_i\|_{L_2(\tilde S_2(\rho,a_i))}\\&\quad+c\varepsilon\sum_{i=1}^2\|w_3\|_{L_6(\tilde S_2(\rho,a_i))}\|\tilde d_i\|_{L_3(\tilde S_2(\rho,a_i))}\left(\int_{\tilde S_2(\rho,a_i)}dx_1\,dx_2\int_r^\rho d\sigma_i\left|\frac{w_3}{\sigma_i}\right|^2\right)^{1/2}\\&\leqslant c\rho^{1/6}\sum_{i=1}^2\|w\|^2_{L_6(\tilde S_2(\rho,a_i))}\|\nabla\tilde d_i\|_{L_2(\tilde S_2(\rho,a_i))}\\&\quad+c\varepsilon\sum_{i=1}^2\|w\|_{L_6(\tilde S_2(\rho,a_i))}\|\nabla w_3\|_{L_2(\tilde S_2(\rho,a_i))}\|\tilde d_i\|_{L_3(\tilde S_2(\rho,a_i))}\\&\leqslant c\big(\rho^{1/6}+\varepsilon\big)\|w\|^2_{H^1(\Omega)}\|\tilde d\|_{W^1_3(\Omega)}.\end{aligned}$$

Thus, we can summarize the estimates for I_1-I_4 to conclude that the nonlinear term in (2.8) is bounded by

$$\left|\int_\Omega (w\cdot\nabla\delta\cdot w+\delta\cdot\nabla w\cdot w)\,dx\right| \leqslant c\|w\|^2_{H^1(\Omega)}\Big(\varepsilon\rho^{\mu-2/3}\sup_{x_3}\|\tilde d\|_{L_3(S_2)}+\rho^{\mu+1/3}\sup_{x_3}\|\tilde d_{,x_3}\|_{L_3(S_2)}$$

$$+\big(\rho^{1/6}+\varepsilon\big)\|\tilde d\|_{W^1_3(\Omega)}+\rho^{1/6}\|\tilde d\|_{H^1(\Omega)}\Big). \tag{2.11}$$

Next, we examine the second term on the r.h.s. of (2.8):

$$\begin{aligned}
\sum_{\alpha=1}^2\|\delta\cdot\bar\tau_\alpha\|^2_{L_2(S_1)} &\leqslant \sum_{\alpha=1}^2\big(\|b\cdot\bar\tau_\alpha\|^2_{L_2(S_1)}+\|\nabla\varphi\cdot\bar\tau_\alpha\|^2_{L_2(S_1)}\big)\\
&\leqslant \|\alpha\|^2_{L_2(S_1)}+c\|\nabla\varphi\|^2_{W^1_{3/2}(\Omega)}\\
&\leqslant \sum_{i=1}^2\|d_i\|^2_{L_2(S_1)}+c\|\operatorname{div} b\|^2_{L_{3/2}(\Omega)}\\
&\leqslant c\|\tilde d\|^2_{W^1_{3/2}(\Omega)}+c\sum_{i=1}^2\big\|\nabla(\tilde d_i\eta_i)\big\|^2_{L_{3/2}(\Omega)}\\
&\leqslant c\|\tilde d\|^2_{W^1_{3/2}(\Omega)}+c\sum_{i=1}^2\big(\|\nabla\tilde d_i\eta_i\|^2_{L_{3/2}(\Omega)}+\|\tilde d_i\nabla\eta_i\|^2_{L_{3/2}(\Omega)}\big)\\
&\leqslant c\|\tilde d\|^2_{W^1_{3/2}(\Omega)}+c\sum_{i=1}^2\|\tilde d_i\nabla\eta_i\|^2_{L_{3/2}(\Omega)}.
\end{aligned}$$

We estimate the last expression in more detail:

$$\begin{aligned}
\sum_{i=1}^2\|\tilde d_i\nabla\eta_i\|^2_{L_{3/2}(\Omega)} &\leqslant \varepsilon^2\Bigg[\Bigg(\int_{-a+r}^{-a+\rho}dx_3\int_{S_2(a_1)}dx'\left|\frac{\tilde d_1}{a+x_3}\right|^{3/2}\Bigg)^{4/3}\\
&\quad+\Bigg(\int_{a-\rho}^{a-r}dx_3\int_{S_2(a_2)}dx'\left|\frac{\tilde d_2}{a-x_3}\right|^{3/2}\Bigg)^{4/3}\Bigg]\\
&\leqslant \varepsilon^2\Bigg[\sup_{x_3}\|\tilde d_1\|^2_{L_{3/2}(S_2(a_1))}\Bigg(\int_{-a+r}^{-a+\rho}\left|\frac{1}{a+x_3}\right|^{3/2}dx_3\Bigg)^{4/3}\\
&\quad+\sup_{x_3}\|\tilde d_2\|^2_{L_{3/2}(S_2(a_2))}\Bigg(\int_{a-\rho}^{a-r}\left|\frac{1}{a-x_3}\right|^{3/2}dx_3\Bigg)^{4/3}\Bigg]\\
&\leqslant c\varepsilon^2\sup_{x_3}\|\tilde d\|^2_{L_{3/2}(S_2)}\Bigg(\int_r^\rho\frac{dy}{y^{3/2}}\Bigg)^{4/3}\leqslant c\varepsilon^2\sup_{x_3}\|\tilde d\|^2_{L_{3/2}(S_2)}\left[\frac{1}{r^{1/2}}-\frac{1}{\rho^{1/2}}\right]^{4/3}\\
&\leqslant c\varepsilon^2\sup_{x_3}\|\tilde d\|^2_{L_{3/2}(S_2)}\frac{1}{\rho^{2/3}}\big[e^{1/2\varepsilon}-1\big]^{4/3}\leqslant c\frac{\varepsilon^2}{\rho^{2/3}}e^{2/3\varepsilon}\sup_{x_3}\|\tilde d\|^2_{L_{3/2}(S_2)}.
\end{aligned}$$

Combining the inequalities above, we infer

$$\sum_{\alpha=1}^{2} \|\delta \cdot \bar{\tau}_\alpha\|^2_{L_2(S_1)} \leqslant c\|\tilde{d}\|^2_{W^1_{3/2}(\Omega)} + c\frac{\varepsilon^2}{\rho^{2/3}} e^{2/3\varepsilon} \sup_{x_3} \|\tilde{d}\|^2_{L_{3/2}(S_2)}.$$

We also estimate the term

$$\begin{aligned}
\|\mathbb{D}(\delta)\|^2_{L_2(\Omega)} &\leqslant \|\mathbb{D}(b)\|^2_{L_2(\Omega)} + \|\mathbb{D}(\nabla\varphi)\|^2_{L_2(\Omega)} \\
&\leqslant \sum_{i=1}^{2}\big(\|\nabla\tilde{d}_i\eta_i\|^2_{L_2(\Omega)} + \|\tilde{d}_i\nabla\eta_i\|^2_{L_2(\Omega)}\big) + \|\nabla^2\varphi\|^2_{L_2(\Omega)} \\
&\leqslant \sum_{i=1}^{2}\big(\|\nabla\tilde{d}_i\eta_i\|^2_{L_2(\Omega)} + \|\tilde{d}_i\nabla\eta_i\|^2_{L_2(\Omega)}\big) + \|\operatorname{div} b\|^2_{L_2(\Omega)} \\
&\leqslant c\sum_{i=1}^{2}\big(\|\nabla\tilde{d}_i\eta_i\|^2_{L_2(\Omega)} + \|\tilde{d}_i\nabla\eta_i\|^2_{L_2(\Omega)}\big) \\
&\leqslant c\sum_{i=1}^{2}\|\tilde{d}_i\|^2_{W^1_2(\Omega)} + \varepsilon^2 c\int_{-a+r}^{-a+\rho} dx_3 \int_{S_2(a_1)} dx' \left|\frac{\tilde{d}_1}{a+x_3}\right|^2 + \varepsilon^2\int_{a-\rho}^{a-r} dx_3 \int_{S_2(a_2)} dx' \left|\frac{\tilde{d}_2}{a-x_3}\right|^2 \\
&\leqslant c\sum_{i=1}^{2}\left(\|\tilde{d}_i\|^2_{W^1_2(\Omega)} + \varepsilon^2 \sup_{x_3}\|\tilde{d}_i\|^2_{L_2(S_2)} \int_r^\rho \frac{dy}{y^2}\right) \\
&\leqslant c\sum_{i=1}^{2}\left[\|\tilde{d}_i\|^2_{W^1_2(\Omega)} + \varepsilon^2 \sup_{x_3}\|\tilde{d}_i\|^2_{L_2(S_2)} \left(\frac{1}{r} - \frac{1}{\rho}\right)\right] \\
&\leqslant c\sum_{i=1}^{2}\left[\|\tilde{d}_i\|^2_{W^1_2(\Omega)} + \varepsilon^2 \sup_{x_3}\|\tilde{d}_i\|^2_{L_2(S_2)} \frac{1}{\rho}\big(e^{1/\varepsilon} - 1\big)\right] \\
&\leqslant c\sum_{i=1}^{2}\left[\|\tilde{d}_i\|^2_{W^1_2(\Omega)} + \frac{\varepsilon^2}{\rho} e^{1/\varepsilon} \sup_{x_3}\|\tilde{d}_i\|^2_{L_2(S_2)}\right].
\end{aligned}$$

Analyzing the last integral on the r.h.s. of (2.8) we have

$$\int_\Omega (f - \delta_t - \delta\cdot\nabla\delta) w\,dx \leqslant \varepsilon_1\|w\|^2_{L_6(\Omega)} + c(1/\varepsilon_1)\big(\|f\|^2_{L_{6/5}(\Omega)} + \|\delta_t\|^2_{L_{6/5}(\Omega)}\big) + \left|\int_\Omega \delta\cdot\nabla\delta\cdot w\,dx\right|.$$

We estimate $\|\delta_t\|_{L_{6/5}(\Omega)}$ as follows

$$\begin{aligned}
\|\delta_t\|_{L_{6/5}(\Omega)} = \|b_t + \nabla\varphi_t\|_{L_{6/5}(\Omega)} &\leqslant \|\tilde{d}_t\|_{L_{6/5}(\Omega)} + \|\operatorname{div} b_t\|_{L_{6/5}(\Omega)} \\
&\leqslant \|\tilde{d}_t\|_{L_{6/5}(\Omega)} + \|\nabla\tilde{d}_t\|_{L_{6/5}(\Omega)} + \|\tilde{d}_t\nabla\eta\|_{L_{6/5}(\Omega)}
\end{aligned}$$

$$\leqslant \|\tilde{d}_t\|_{W^1_{6/5}(\Omega)} + \varepsilon \sup_{x_3} \|\tilde{d}_t\|_{L_{6/5}(S_2)} \left(\int_r^\rho \frac{dx_3}{x_3^{6/5}} \right)^{5/6}$$
$$\leqslant \|\tilde{d}_t\|_{W^1_{6/5}(\Omega)} + \varepsilon \frac{1}{\rho^{1/6}} e^{1/6\varepsilon} \sup_{x_3} \|\tilde{d}_t\|_{L_{6/5}(S_2)}$$

since

$$\left(\int_r^\rho \frac{dx_3}{x_3^{6/5}} \right)^{5/6} = \left(\frac{1}{r^{1/5}} - \frac{1}{\rho^{1/5}} \right)^{5/6} = \frac{1}{\rho^{1/6}} \left(e^{1/5\varepsilon} - 1\right)^{5/6}.$$

Finally, we examine

$$\left| \int_\Omega \delta \cdot \nabla \delta \cdot w \, dx \right| \leqslant \|\nabla \delta\|_{L_2(\Omega)} \|w\|_{L_6(\Omega)} \|\delta\|_{L_3(\Omega)} \leqslant \varepsilon_2 \|w\|^2_{L_6(\Omega)} + c(1/\varepsilon_2) \|\delta\|^4_{W^1_2(\Omega)}$$
$$\leqslant \varepsilon_2 \|w\|^2_{L_6(\Omega)} + c(1/\varepsilon_2) \left(\|\tilde{d}\|^4_{W^1_2(\Omega)} + \frac{\varepsilon^4}{\rho^2} e^{2/\varepsilon} \sup_{x_3} \|\tilde{d}\|^4_{L_2(S_2)} \right).$$

We summarize the above estimates to rewrite (2.8) as follows

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|w\|^2_{L_2(\Omega)} + \nu \|w\|^2_{H^1(\Omega)} + \gamma \sum_{\alpha=1}^{2} \|w \cdot \bar{\tau}_\alpha\|^2_{L_2(S_1)} \\
&\leqslant \|w\|^2_{H^1(\Omega)} \Big[\varepsilon \rho^{\mu - 2/3} \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + \rho^{\mu + 1/3} \sup_{x_3} \|\tilde{d}_{,x_3}\|_{L_3(S_2)} \\
&\quad + \left(\rho^{1/6} + \varepsilon\right) \|\tilde{d}\|_{W^1_3(\Omega)} + \rho^{1/6} \|\tilde{d}\|_{H^1(\Omega)} + \varepsilon_1 + \varepsilon_2 \Big] + \|f\|^2_{L_{6/5}(\Omega)} + \|\tilde{d}\|^2_{L_2(\Omega)} + \|\tilde{d}\|^4_{W^1_2(\Omega)} \\
&\quad + \|\tilde{d}\|^2_{W^1_2(\Omega)} + \|\nabla \tilde{d}\|^2_{L_{6/5}(\Omega)} + \|\nabla \tilde{d}\|^4_{L_2(\Omega)} + \|\tilde{d}\|^2_{W^1_{3/2}(\Omega)} + \|\tilde{d}_t\|^2_{W^1_{6/5}(\Omega)} + \frac{\varepsilon^2}{\rho} e^{1/\varepsilon} \sup_{x_3} \|\tilde{d}\|^2_{L_2(S_2)} \\
&\quad + \frac{\varepsilon^4}{\rho^2} e^{2/\varepsilon} \sup_{x_3} \|\tilde{d}\|^4_{L_2(S_2)} + \frac{\varepsilon^2}{\rho^{2/3}} e^{2/3\varepsilon} \sup_{x_3} \|\tilde{d}\|^2_{L_{3/2}(S_2)} + \frac{\varepsilon^2}{\rho^{1/3}} e^{1/3\varepsilon} \sup_{x_3} \|\tilde{d}_t\|^2_{L_{6/5}(S_2)}.
\end{aligned} \tag{2.12}$$

We apply the Sobolev anisotropic imbedding (see [1, Chapter 3, Section 10]) to estimate $\sup_{x_3} \|\tilde{d}\|_{L_3(S_2)}$ and $\sup_{x_3} \|\tilde{d}_{,x_3}\|_{L_3(S_2)}$ by some W^s_p norm and calculate

$$2\left(\frac{1}{p} - \frac{1}{3}\right)\frac{1}{s} + \frac{1}{p} \cdot \frac{1}{s} + \frac{1}{s} \leqslant 1 \quad \text{for } p > 3,$$
$$2\left(\frac{1}{p} - \frac{1}{3}\right)\frac{1}{s} + \frac{1}{p} \cdot \frac{1}{s} + \frac{1}{s} < 1 \quad \text{for } p = 3.$$

Thus,

$$\frac{3}{p} + \frac{1}{3} \leqslant s, \quad p > 3 \text{ or } p = 3,\ s > \frac{4}{3}. \tag{2.13}$$

We set $\mu > \frac{2}{3}$; then since $\rho < 1$, we have $\rho^{\mu+1/3} \leqslant \rho^{1/6}$. Therefore

$$\begin{aligned}
&\varepsilon\rho^{\mu-2/3} \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + \rho^{\mu+1/3} \sup_{x_3} \|\nabla\tilde{d}\|_{L_3(S_2)} + \big(\rho^{1/6}+\varepsilon\big)\|\tilde{d}\|_{W_3^1(\Omega)} + \rho^{1/6}\|\tilde{d}\|_{H^1(\Omega)} \\
&\quad \leqslant \big(\varepsilon\rho^{\mu-2/3} + \rho^{\mu+1/3} + 2\rho^{1/6} + \varepsilon\big)\|\tilde{d}\|_{W_p^s(\Omega)} \leqslant \big(2\varepsilon + 3\rho^{1/6}\big)\|\tilde{d}\|_{W_p^s(\Omega)}.
\end{aligned}$$

We put

$$\begin{aligned}
\varepsilon &= \frac{\nu}{15\|\tilde{d}\|_{W_p^s(\Omega)}}, \\
\rho^{1/6} &= \frac{\nu}{15\|\tilde{d}\|_{W_p^s(\Omega)}}, \\
\varepsilon_1 + \varepsilon_2 &= \frac{\nu}{6},
\end{aligned} \tag{2.14}$$

with p, s satisfying (2.13). Then,

$$\begin{aligned}
&\varepsilon\rho^{\mu-2/3} \sup_{x_3} \|\tilde{d}\|_{L_3(S_2)} + \rho^{\mu+1/3} \sup_{x_3} \|\nabla\tilde{d}\|_{L_3(S_2)} + \big(\rho^{1/6}+\varepsilon\big)\|\tilde{d}\|_{W_3^1(\Omega)} \\
&\quad + \rho^{1/6}\|\tilde{d}\|_{H^1(\Omega)} + \varepsilon_1 + \varepsilon_2 \leqslant \frac{\nu}{2}
\end{aligned}$$

and formula (2.12) assumes the form

$$\begin{aligned}
&\frac{d}{dt}\|w\|_{L_2(\Omega)}^2 + \nu\|w\|_{H^1(\Omega)}^2 + \gamma \sum_{\alpha=1}^2 \|w \cdot \bar{\tau}_\alpha\|_{L_2(S_1)}^2 \\
&\quad \leqslant 2\|f\|_{L_{6/5}(\Omega)}^2 + \varphi\big(\|\tilde{d}\|_{W_2^1(\Omega)}\big)\big(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2\big) \\
&\qquad + \varphi\big(\|\tilde{d}\|_{W_p^s(\Omega)}\big)\Big(\sup_{x_3}\|\tilde{d}\|_{L_2(S_2)}^2 + \sup_{x_3}\|\tilde{d}_t\|_{L_{6/5}(S_2)}^2\Big)
\end{aligned}$$

where φ is a nonlinear positive increasing function. We use the Sobolev imbeddings

$$\begin{aligned}
\sup_{x_3}\|\tilde{d}\|_{L_2(S_2)} &\leqslant c\|\tilde{d}\|_{W_2^1(\Omega)}, \\
\sup_{x_3}\|\tilde{d}_t\|_{L_{6/5}(S_2)} &\leqslant c\|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{d}{dt}\|w\|_{L_2(\Omega)}^2 + \nu\|w\|_{H^1(\Omega)}^2 + \gamma \sum_{\alpha=1}^2 \|w \cdot \bar{\tau}_\alpha\|_{L_2(S_1)}^2 \\
&\quad \leqslant 2\|f\|_{L_{6/5}(\Omega)}^2 + \varphi\big(\|\tilde{d}\|_{W_p^s(\Omega)}\big)\big(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2\big).
\end{aligned} \tag{2.15}$$

Integrating (2.15) with respect to time we obtain

$$
\begin{aligned}
&\|w\|^2_{V_2^0(\Omega^t)} + \gamma \sum_{\alpha=1}^{2} \int_0^t \|w\cdot\bar\tau_\alpha\|^2_{L_2(S_1)}\,dt \\
&\quad\leqslant 2\|f\|^2_{L_2(0,t;L_{6/5}(\Omega))} + \varphi\Big(\sup_\tau \|\tilde d\|_{W_p^s(\Omega)}\Big)\big(\|\tilde d\|^2_{L_2(0,t;W_2^1(\Omega))} + \|\tilde d_t\|^2_{L_2(0,t;W^1_{6/5}(\Omega))}\big) \\
&\qquad + \big\|w(0)\big\|^2_{L_2(\Omega)},
\end{aligned}
\tag{2.16}
$$

where $\frac{3}{p}+\frac{1}{3}\leqslant s$, $p>3$ or $p=3$, $s>\frac{4}{3}$. $\square$

3. Weak solutions to (2.6)

In this section, we use the Galerkin method to prove the existence of weak solutions to the problem (2.6). We follow the ideas of [5, Chapter 6, Section 7]. Namely, we introduce a sequence of approximating functions w^N given as

$$
w^N(x,t) = \sum_{k=1}^{N} C_{kN}(t)a^k(x),
$$

where $\{a^k\}_{k=1}^\infty$ is a system of orthonormal functions in $L^2(\Omega)\cap J_2^0(\Omega)$. Here,

$$
J_2^0(\Omega) = \big\{f\in H^1(\Omega)\colon \operatorname{div} f = 0\big\}
$$

and $\{a^k\}_{k=1}^\infty$ is a fundamental system in $H^1(\Omega)$ with $\sup_{x\in\Omega}|a^k(x)|<\infty$, $\sup_{x\in\partial\Omega}|a^k(x)|<\infty$. The coefficients $C_{kN}(0)$ are defined by

$$
C_{kN}|_{t=0} = (w_0, a_k), \quad k=1,\dots,N,
$$

and the functions w^N satisfy the following system with test functions a^k:

$$
\begin{aligned}
&\bigg\{ \int_\Omega \bigg(\frac{1}{2}\frac{d}{dt} w^N a^k + w^N\cdot\nabla w^N a^k + \delta\cdot\nabla w^N\cdot a^k + w^N\cdot\nabla\delta\cdot a^k + \nu\mathbb{D}\big(w^N\big)\mathbb{D}\big(a^k\big)\bigg)\,dx \\
&\quad + \gamma\int_{S_1} w^N\cdot\bar\tau_j a^k\bar\tau_j\,dS_1\bigg\} = \bigg(\sum_{j,\sigma=1}^{2}\int_{S_\sigma} B_{\sigma j}a^k\cdot\bar\tau_j\,dS_\sigma + \int_\Omega F\cdot a^k\,dx\bigg)
\end{aligned}
$$

for $k=1,\dots,N$. Thus, w^N are weak solutions to (2.6).

With $(f,g)=\int_\Omega fg\,dx$ and $(f,g)_S=\int_S fg\,dS$ this can be rewritten as

$$
\begin{aligned}
&\big\{\big(w_t^N,a^k\big) + \big(w^N\cdot\nabla w^N,a^k\big) + \big(\delta\cdot\nabla w^N,a^k\big) + \big(w^N\cdot\nabla\delta,a^k\big) \\
&\quad + \nu\big(\mathbb{D}\big(w^N\big),\mathbb{D}\big(a^k\big)\big) + \gamma\big(w^N\cdot\bar\tau_j,a^k\cdot\bar\tau_j\big)_{S_1}\big\} \\
&= \Bigg[\sum_{\sigma,j=1}^{2}\big(B_{\sigma j},a^k\cdot\bar\tau_j\big)_{S_\sigma} + \big(F,a^k\big)\Bigg], \quad k=1,\dots,N.
\end{aligned}
$$

Thus,

$$\begin{aligned}&\left(\frac{d}{dt}w^N, a^k\right) + \left(w^N\cdot\nabla w^N, a^k\right) + \left(\delta\cdot\nabla w^N, a^k\right) + \left(w^N\cdot\nabla\delta, a^k\right)\\&\quad + \nu\left(\mathbb{D}\left(w^N\right), \mathbb{D}\left(a^k\right)\right) + \gamma\left(w^N\cdot\bar\tau_j, a^k\cdot\bar\tau_j\right)_{S_1}\\&= \sum_{j,\sigma=1}^{2}\left(B_{\sigma j}, a^k\cdot\bar\tau_j\right)_{S_\sigma} + \left(F, a^k\right), \quad k=1,\dots,N. \end{aligned}\tag{3.1}$$

The above equations are in fact a system of ordinary differential equations for the functions $C_{kN}(t)$. The properties of the sequence a^k imply

$$\left\|w^N(x,t)\right\|^2_{L_2(\Omega)} = \sum_{k=1}^{N} C^2_{kN}(t).$$

On the other hand, we can obtain a priori bounds for the approximate solutions w^N of the same form as (2.16):

$$\begin{aligned}\left\|w^N\right\|^2_{V_2^0(\Omega^T)} &= \sup_{0\leqslant t\leqslant T}\left\|w^N\right\|_{L_2(\Omega)} + \int_0^T\left\|\nabla w^N\right\|_{L_2(\Omega)}dt\\&\leqslant \int_0^T\|f\|^2_{L_{6/5}(\Omega)} + \varphi\left(\sup_{0\leqslant t\leqslant T}\|\tilde d\|_{W_p^s(\Omega)}\right)\int_0^T\left(\|\tilde d\|^2_{W_2^1(\Omega)} + \|\tilde d_t\|^2_{W^1_{6/5}(\Omega)}\right)dt\\&\quad + \left\|w^N(0)\right\|^2_{L_2(\Omega)}\\&\leqslant C,\end{aligned}\tag{3.2}$$

where $\frac{3}{p}+\frac{1}{3}\leqslant s$, $p>3$ or $p=3$, $s>\frac{4}{3}$. Therefore, $\sup_{0\leqslant t\leqslant T}|C_{kN}(t)|$ is bounded on $[0,T]$ and w^N are well defined for all times t.

Let us now define $\psi_{N,k}\equiv(w^N(x,t),a^k(x))$. This sequence is uniformly bounded by (3.2). We can also show that it is equicontinuous. Namely, we integrate (3.1) with respect to t from t to $t+\Delta t$ to obtain

$$\begin{aligned}&\left|\psi_{N,k}(t+\Delta t)-\psi_{N,k}(t)\right|\\&\quad\leqslant \sup_{x\in\Omega}\left|a^k(x)\right|\int_t^{t+\Delta t}\left(\left|w^N\cdot\nabla w^N\right|_{L_2(\Omega)} + \left|\delta\cdot\nabla w^N\right|_{L_2(\Omega)} + \left|w^N\cdot\nabla\delta\right|_{L_2(\Omega)} + |F|_{L_2(\Omega)}\right)d\tau\\&\qquad + \nu\left|\nabla a^k\right|_{L_2(\Omega)}\int_t^{t+\Delta t}\left|\nabla w^N\right|_{L_2(\Omega)}d\tau + \gamma\sup_{x\in S}\left|a^k(x)\right|\int_t^{t+\Delta t}\left(\left|w^N\cdot\bar\tau_j\right|_{L_2(S_1)} + \sum_{j,\sigma=1}^{2}|B_{\sigma j}|_{L_2(S_\sigma)}\right)d\tau\\&\quad\leqslant \sup_{x\in\Omega}\left|a^k(x)\right|\sqrt{\Delta t}\left(\sup_{x\in\Omega}\left|w^N\right|_{L_2(\Omega)}\left(\left|\nabla w^N\right|_{L_2(\Omega^T)} + |\nabla\delta|_{L_2(\Omega^T)}\right) + \sup_{x\in\Omega}|\delta|_{L_2(\Omega)}\left|\nabla w^N\right|_{L_2(\Omega^T)}\right)\\&\qquad + \sup_{x\in\Omega}\left|a^k(x)\right|\int_t^{t+\Delta t}|F|_{L_2(\Omega)}d\tau + \nu\left|\nabla a^k\right|_{L_2(\Omega)}\sqrt{\Delta t}\left|\nabla w^N\right|_{L_2(\Omega^T)}\end{aligned}$$

$$+\gamma \sup_{x\in S}\left|a^k(x)\right|\left(\sqrt{\Delta t}\left|\nabla w^N\right|_{L_2(\Omega^T)}+\int_t^{t+\Delta t}\sum_{j=1}^2|B_j|_{L_2(S)}\,d\tau\right)$$

$$\leqslant C(k)\left(\sqrt{\Delta t}+\int_t^{t+\Delta t}\left(|F|_{L_2(\Omega)}+\sum_{j=1}^2|B_j|_{L_2(S)}\right)d\tau\right).$$

We can see that for given k and $N \geqslant k$ the r.h.s. tends to zero as $\Delta t \to 0$ uniformly in N. Thus, it is possible to choose a subsequence N_m such that $\psi_{N_m,k}$ converges as $m\to\infty$ uniformly to some continuous function ψ_k for any given k. Since the limit function w is defined as

$$w(x,t)=\sum_{k=1}^\infty \psi_k(t)a^k(x),$$

we conclude that $(w^{N_m}-w,\psi(x))$ tends to zero as $m\to\infty$ uniformly with respect to $t\in[0,T]$ for any $\psi\in J_2^0(\Omega)$ and $w(x,t)$ is continuous in t in weak topology. Moreover, estimates (3.2) remain true for the limit function w.

We will show that $\{w^{N_m}\}$ converges strongly in $L_2(\Omega^T)$. To this end, we need to apply the following version of the Friedrichs lemma: for any $\varepsilon>0$, there exists N_ε such that for any $u\in W_2^1(\Omega)$ the following inequality holds

$$\|u\|^2_{L_2(\Omega)}\leqslant\sum_{k=1}^{N_\varepsilon}\left(u,a^k\right)+\varepsilon\|\nabla u\|^2_{L_2(\Omega)}.$$

This in terms of $u=w^{N_m}-w^{N_l}$ reads

$$\left\|w^{N_m}-w^{N_l}\right\|^2_{L_2(\Omega^T)}\leqslant\sum_{k=1}^{N_\varepsilon}\int_0^T\left(w^{N_m}-w^{N_l},a^k\right)dt+\varepsilon\left\|\nabla w^{N_m}-\nabla w^{N_l}\right\|^2_{L_2(\Omega^T)}.$$

By (3.2), we have

$$\left\|\nabla w^{N_m}-\nabla w^{N_l}\right\|^2_{L_2(\Omega^T)}\leqslant 2C^2$$

for some constant C. The first integral on the r.h.s. for a given number N_ε can be arbitrarily small if only m and l are sufficiently large, so it tends to zero as $m,l\to\infty$. Therefore, $\{w^{N_m}\}$ converges strongly in $L_2(\Omega^T)$.

We summarize the above convergence properties of the sequence $\{w^{N_m}\}$:

(i) $\{w^{N_m}\}\to w$ strongly in $L_2(\Omega^T)$ for some w,
(ii) $\{w^{N_m}\}\to w$ weakly in $L_2(\Omega)$ uniformly with respect to $t\in[0,T]$,
(iii) $\nabla\{w^{N_m}\}\to\nabla w$ weakly in $L_2(\Omega^T)$.

For given $\Phi^k=\sum_{j=1}^k d_j(t)a^j(x)$, the sequence $\{w^{N_m}\}$ satisfies the identities

$$\int_\Omega\left(\frac{d}{dt}w^{N_m}\Phi^k+\left(w^{N_m}\cdot\nabla w^{N_m}+\delta\cdot\nabla w^{N_m}+w^{N_m}\cdot\nabla\delta\right)\Phi^k+\nu\mathbb{D}\left(w^{N_m}\right)\mathbb{D}\left(\Phi^k\right)\right)dx$$

$$+\gamma\int_{S_1}w^{N_m}\cdot\bar\tau_j\Phi^k\cdot\bar\tau_j\,dS_0=\sum_{\sigma,j=1}^2\int_{S_\sigma}B_{\sigma j}\Phi^k\cdot\bar\tau_j\,dS_\sigma+\int_\Omega F\Phi^k\,dx.$$

Then, we can pass to the limit with $m \to \infty$ to obtain the identity for w. The conditions $\operatorname{div} w^N = 0$, $w^N \cdot \bar{n}|_{S^T} = 0$ remain true for the limit function w as well.

It remains to consider the limit $\lim_{t\to 0} w(x,t)$. We note that w^{N_m} satisfies the relation (2.8) (if we use the test function w^{N_m}). This yields

$$\left\| w^{N_m} \right\|_{L_2(\Omega)} \leqslant \|w_0\|_{L_2(\Omega)} + \int_0^t \left(\|F\|_{L_2(\Omega)} + \|B\|_{L_2(S)} \right) dt.$$

In the limit $m \to \infty$ we obtain

$$\|w\|_{L_2(\Omega)} \leqslant \|w_0\|_{L_2(\Omega)} + \int_0^t \left(\|F\|_{L_2(\Omega)} + \|B\|_{L_2(S)} \right) dt,$$

which implies

$$\overline{\lim_{t\to 0}} \|w\|_{L_2(\Omega)} \leqslant \|w_0\|_{L_2(\Omega)}.$$

On the other hand, since w^{N_m} tends to w as $m \to \infty$, we have $\|w^{N_m} - w_0\|_{L_2(\Omega)} \to 0$. Therefore, $|w^{N_m} - w_0| \to 0$ weakly in $L^2(\Omega)$ as $t \to 0$ and

$$\|w_0\|_{L_2(\Omega)} \leqslant \underline{\lim}_{t\to 0} \|w\|_{L_2(\Omega)}.$$

We conclude that the limit $\lim_{t\to 0} \|w\|_{L_2(\Omega)}$ exists and is equal to $\|w_0\|_{L_2(\Omega)}$ where the convergence is strong, in the $L_2(\Omega)$ norm.

Consequently, we have proved the following result.

Lemma 3.3. *Let the assumptions of Lemma* 2.2 *be satisfied. Then there exists a weak solution w to problem* (2.6) *such that w is weakly continuous with respect to t in the $L_2(\Omega)$ norm and w converges to w_0 as $t \to 0$ strongly in the $L_2(\Omega)$ norm.*

Since $v = w - \delta$ we deduce the analogous existence result for v formulated in Theorem 1.

4. Global solutions to (2.6)

To obtain a global estimate we write (2.7) in the form

$$\frac{d}{dt}\|w\|_{L_2(\Omega)}^2 + \nu\|w\|_{L_2(\Omega)}^2 \leqslant 2\|f\|_{L_{6/5}(\Omega)}^2 + \varphi\left(\|\tilde{d}\|_{W_p^s(\Omega)}\right)\left(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2\right),$$

where $\frac{3}{p} + \frac{1}{3} \leqslant s$, $p > 3$ or $p = 3$, $s > \frac{4}{3}$. Hence

$$\frac{d}{dt}\left(\|w\|_{L_2(\Omega)}^2 e^{\nu t}\right) \leqslant 2\|f\|_{L_{6/5}(\Omega)}^2 e^{\nu t} + \varphi\left(\|\tilde{d}\|_{W_p^s(\Omega)}\right)\left(\|\tilde{d}\|_{W_2^1(\Omega)}^2 + \|\tilde{d}_t\|_{W_{6/5}^1(\Omega)}^2\right)e^{\nu t}.$$

Integrating with respect to time from t_1 to t_2 yields

$$\left\|w(t_2)\right\|^2_{L_2(\Omega)} e^{\nu t_2} \leqslant 2\int_{t_1}^{t_2} \|f\|^2_{L_{6/5}(\Omega)} e^{\nu t}\,dt + \left\|w(t_1)\right\|^2_{L_2(\Omega)} e^{\nu t_1}$$
$$+ \varphi\Big(\sup_t \|\tilde d\|_{W^s_p(\Omega)}\Big) \int_{t_1}^{t_2} \big(\|\tilde d\|^2_{W^1_2(\Omega)} + \|\tilde d_t\|^2_{W^1_{6/5}(\Omega)}\big) e^{\nu t}\,dt.$$

Thus,

$$\left\|w(t_2)\right\|^2_{L_2(\Omega)} \leqslant 2e^{-\nu t_2}\int_{t_1}^{t_2} \|f\|^2_{L_{6/5}(\Omega)} e^{\nu t}\,dt + \left\|w(t_1)\right\|^2_{L_2(\Omega)} e^{-\nu(t_2-t_1)}$$
$$+ \varphi\Big(\sup_t \|\tilde d\|_{W^s_p(\Omega)}\Big) e^{-\nu t_2} \int_{t_1}^{t_2} \big(\|\tilde d\|^2_{W^1_2(\Omega)} + \|\tilde d_t\|^2_{W^1_{6/5}(\Omega)}\big) e^{\nu t}\,dt$$

and this implies

$$\left\|w(t_2)\right\|^2_{L_2(\Omega)} \leqslant 2\int_{t_1}^{t_2} \|f\|^2_{L_{6/5}(\Omega)}\,dt + \left\|w(t_1)\right\|^2_{L_2(\Omega)} e^{-\nu(t_2-t_1)}$$
$$+ \varphi\Big(\sup_t \|\tilde d\|_{W^s_p(\Omega)}\Big) \int_{t_1}^{t_2} \big(\|\tilde d\|^2_{W^1_2(\Omega)} + \|\tilde d_t\|^2_{W^1_{6/5}(\Omega)}\big)\,dt. \tag{4.1}$$

Setting $t_1 = 0$ and $t_2 = t \in \mathbb{R}_+$ we obtain the global estimate

$$\left\|w(t)\right\|^2_{L_2(\Omega)} \leqslant 2\int_0^t \|f\|^2_{L_{6/5}(\Omega)}\,d\tau + \left\|w(0)\right\|^2_{L_2(\Omega)} e^{-\nu t}$$
$$+ \varphi\Big(\sup_t \|\tilde d\|_{W^s_p(\Omega)}\Big) \int_0^t \big(\|\tilde d\|^2_{W^1_2(\Omega)} + \|\tilde d_t\|^2_{W^1_{6/5}(\Omega)}\big)\,d\tau. \tag{4.2}$$

Let $k \in \mathbb{N}$. Integrating (2.7) with respect to time from kT to $t \in (kT, (k+1)T]$ we get

$$\|w\|^2_{V^0_2(\Omega\times(kT,t))} \leqslant 2\int_{kT}^t \|f\|^2_{L_{6/5}(\Omega)}\,d\tau + \left\|w(kT)\right\|^2_{L_2(\Omega)}$$
$$+ \varphi\Big(\sup_\tau \|\tilde d\|_{W^s_p(\Omega)}\Big) \int_{kT}^t \big(\|\tilde d\|^2_{W^1_2(\Omega)} + \|\tilde d_t\|^2_{W^1_{6/5}(\Omega)}\big)\,d\tau. \tag{4.3}$$

Therefore,

$$\|v\|^2_{V_2^0(\Omega\times(kT,t))} \leqslant 2\int_{kT}^{t} \|f\|^2_{L_{6/5}(\Omega)}\,d\tau + \big\|v(kT)\big\|^2_{L_2(\Omega)} + \varphi\Big(\sup_{\tau}\|\tilde d\|_{W_p^s(\Omega)}\Big)\int_{kT}^{t}\big(\|\tilde d\|^2_{W_2^1(\Omega)} + \|\tilde d_t\|^2_{W^1_{6/5}(\Omega)}\big)\,d\tau. \tag{4.4}$$

We also have

$$\big\|v(T)\big\|^2_{L_2(\Omega)} \leqslant 2\int_{0}^{t} \|f\|^2_{L_{6/5}(\Omega)}\,d\tau + \big\|v(0)\big\|^2_{L_2(\Omega)}e^{-\nu T} + \varphi\Big(\sup_{t}\|\tilde d\|_{W_p^s(\Omega)}\Big)\int_{0}^{t}\big(\|\tilde d\|^2_{W_2^1(\Omega)} + \|\tilde d_t\|^2_{W^1_{6/5}(\Omega)}\big)\,d\tau. \tag{4.5}$$

We set $\mu_1 = e^{-\nu T}$. Let us assume that

$$\big\|v(0)\big\|_{L_2(\Omega)} \leqslant A$$

for some constant A and

$$2\int_0^t \|f\|^2_{L_{6/5}(\Omega)}\,d\tau + \varphi\Big(\sup_t\|\tilde d\|_{W_p^s(\Omega)}\Big)\int_0^t\big(\|\tilde d\|^2_{W_2^1(\Omega)} + \|\tilde d_t\|^2_{W^1_{6/5}(\Omega)}\big)\,d\tau \leqslant \big(1-e^{-\nu T}\big)A^2.$$

Thus,

$$\big\|v(T)\big\|_{L_2(\Omega)} \leqslant A$$

so we can control the initial condition for the next time step. This can be repeated for intervals $(kT, (k+1)T)$. Then by (4.4) we can prove global existence of a weak solution such that

$$v \in V_2^0\big(\Omega \times \big(kT, (k+1)T\big)\big), \quad \forall k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\},$$

proving Theorem 2.

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